Pergamon

Computers Math. Applic. Vol. 32, No. 6, pp. 95–99, 1996

0898-1221/96 $15.00 + 0.00

PII: S0898-1221(96)00146-0

An Application of Hayashi's Inequality for Differentiable Functions

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(Received April 1996; accepted May 1996)

Abstract—In this paper, we offer an application of Hayashi's generalization of Steffensen's inequality for differentiable functions, as well as its connections with the celebrated Hermite-Hadamard's integral inequality for convex functions. Some applications for special means are also given.

Keywords—Hayashi's inequality, Hermite-Hadamard's inequality, Steffensen's inequality, Integral inequalities, Special means.

1. THE MAIN RESULT

To prove our main result, we shall need the following generalization of the well-known Steffensen's inequality which is due to Hayashi [1, pp. 311–312].

THEOREM 1. *Let $f : [a,b] \to \mathbb{R}$ be a nonincreasing mapping on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ an integrable mapping on $[a,b]$ with*

$$0 \le g(x) \le A, \qquad \text{for all } x \in [a,b].$$

Then, the following inequality holds

$$A\int_{b-\lambda}^{b} f(x)\,dx \le \int_a^b f(x)g(x)\,dx \le A\int_a^{a+\lambda} f(x)\,dx, \tag{1}$$

where

$$\lambda = \frac{1}{A}\int_a^b g(x)\,dx.$$

THEOREM 2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $[a,b] \subset I^\circ$ with $M = \sup_{x\in[a,b]} f'(x) < \infty$, $m = \inf_{x\in[a,b]} f'(x) > -\infty$ and $M > m$. If f' is integrable on $[a,b]$, then the following inequality holds*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2}\right| \le \frac{[f(b)-f(a)-m(b-a)][M(b-a)-f(b)+f(a)]}{2(M-m)(b-a)} \tag{2}$$

$$\le \frac{(M-m)(b-a)}{8}. \tag{3}$$

PROOF. Let $f(x) = a - x$ and $g(x) = f'(x) - m$, and apply Hayashi's inequality, to obtain

$$(M-m)\int_{b-\lambda}^{b} (a-x)\,dx \le I \le (M-m)\int_a^{a+\lambda} (a-x)\,dx, \tag{4}$$

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX where

$$I=\int_a^b (a-x)(f'(x)-m)\,dx,$$

and

$$\lambda=\frac{1}{M-m}\int_a^b (f'(x)-m)\,dx=\frac{f(b)-f(a)-m(b-a)}{M-m}.$$

Since,

$$\int_{b-\lambda}^b (a-x)\,dx=\frac{1}{2}\left[(b-a-\lambda)^2-(b-a)^2\right],$$

and

$$\int_a^{a+\lambda}(a-x)\,dx=-\frac{\lambda^2}{2},$$

the inequality (4) is the same as

$$\ell_1\equiv(M-m)\left[\frac{(b-a-\lambda)^2-(b-a)^2}{2}\right]\le I\le(M-m)\left[-\frac{\lambda^2}{2}\right]\equiv\ell_2.$$

Next, since

$$\begin{aligned}\frac{\ell_1+\ell_2}{2}&=\frac{M-m}{2}\left[-\frac{\lambda^2}{2}+\frac{(b-a-\lambda)^2}{2}-\frac{(b-a)^2}{2}\right]\\&=\frac{(M-m)[-\lambda(b-a)]}{2}=\frac{m(b-a)^2}{2}-\frac{(b-a)(f(b)-f(a))}{2},\end{aligned}$$

and

$$I=\int_a^b f(x)\,dx-(b-a)f(b)+m\frac{(b-a)^2}{2},$$

it follows that

$$\begin{aligned}\left|I-\frac{\ell_1+\ell_2}{2}\right|&=\left|\int_a^b f(x)\,dx-(b-a)f(b)+m\frac{(b-a)^2}{2}-m\frac{(b-a)^2}{2}+\frac{(b-a)(f(b)-f(a))}{2}\right|\\&=\left|\int_a^b f(x)\,dx-(b-a)\frac{f(a)+f(b)}{2}\right|.\end{aligned}\tag{5}$$

We also have

$$\begin{aligned}\left|I-\frac{\ell_1+\ell_2}{2}\right|&\le\frac{\ell_2-\ell_1}{2}\\&=\frac{M-m}{2}\left[-\frac{\lambda^2}{2}+\frac{(b-a)^2}{2}-\frac{(b-a-\lambda)^2}{2}\right]\\&=\frac{M-m}{2}\left[-\lambda^2+(b-a)\lambda\right]\\&=\frac{[f(b)-f(a)-m(b-a)][M(b-a)-f(b)+f(a)]}{2(M-m)}.\end{aligned}\tag{6}$$

Now on combining (5) and (6) we immediately obtain the inequality (2).

To prove (3), we define the mapping $g:\mathbb{R}\to\mathbb{R}, g(t)=-t^2+(b-a)t$. It is clear that

$$g(t)\le g\left(\frac{b-a}{2}\right)=\frac{(b-a)^2}{4},\qquad\text{for all } t\in\mathbb{R}.\tag{7}$$

We choose

$$t = \lambda = \frac{f(b) - f(a) - m(b-a)}{M - m},$$

so that

$$\frac{M-m}{2} g(\lambda) = \frac{[f(b) - f(a) - m(b-a)][M(b-a) - f(b) + f(a)]}{2(M-m)}. \tag{8}$$

From (7) and (8), we have

$$\frac{M-m}{2} g(\lambda) \le \frac{(M-m)(b-a)^2}{8},$$

which in view of (2) proves the required inequality (3).

COROLLARY 3. *Let the conditions of Theorem 2 be satisfied, and* $\|f'\|_\infty = \sup_{x\in[a,b]} |f'(x)| < \infty$. *Then, the following inequality holds*

$$\begin{aligned}\left| \frac{1}{b-a} \int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2} \right| &\le \frac{\|f'\|_\infty^2 \left[(b-a)^2 - (f(b)-f(a))^2\right]}{4\|f'\|_\infty (b-a)} \\ &\le \frac{\|f'\|_\infty (b-a)}{4}.\end{aligned}$$

PROOF. The corollary follows from Theorem 2 by choosing $m = -\|f'\|_\infty$ and $M = \|f'\|_\infty$.

REMARK 1. The first inequality in Corollary 3 was proved by Lyengar in 1938 (see [2, p. 471]) by means of geometrical arguments.

The case of convex mappings is embodied in the following corollary, which is very important in applications.

COROLLARY 4. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable convex mapping on* I^o *and* $[a,b] \subset I^o$ *with* $f'(b) \neq f'(a)$. *Then, the following inequality holds*

$$\begin{aligned} 0 &\le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \\ &\le \frac{[f(b) - f(a) - f'(a)(b-a)]\,[f'(b)(b-a) - f(b) + f(a)]}{2\,(f'(b) - f'(a))\,(b-a)} \\ &\le \frac{(f'(b) - f'(a))\,(b-a)}{8}. \end{aligned}$$

PROOF. The corollary follows from Theorem 2 and the observation that we can choose $m = f'(a)$ and $M = f'(b)$.

REMARK 2. For a convex mapping $f : [a,b] \to \mathbb{R}$, the double inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}$$

is well known in literature as Hermite-Hadamard's integral inequality [3] and references therein. The result of Corollary 3 gives a counterpart of the above inequality. Indeed, we have

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \ge 0,$$

when the mapping f is differentiable on I^o, i.e., in the usual cases which are important in applications.

2. APPLICATIONS TO SPECIAL MEANS

For $\alpha,\ \beta > 0$ we recall the means

$$A(\alpha,\beta) = \frac{\alpha+\beta}{2}, \qquad \text{arithmetic mean,}$$
$$G(\alpha,\beta) = \sqrt{\alpha\beta}, \qquad \text{geometric mean,}$$
$$H(\alpha,\beta) = \frac{2}{(1/\alpha)+(1/\beta)}, \qquad \text{harmonic mean,}$$
$$L(\alpha,\beta) = \frac{\beta-\alpha}{\ln\beta-\ln\alpha}, \qquad \text{logarithmic mean } (\alpha \neq \beta),$$
$$I(\alpha,\beta) = \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right)^{1/(\beta-\alpha)}, \qquad \text{identric mean } (\alpha \neq \beta),$$
$$L_p(\alpha,\beta) = \left[\frac{\beta^{p+1}-\alpha^{p+1}}{(p+1)(\beta-\alpha)}\right]^{1/p}, \qquad \text{generalized log-mean } (\alpha \neq \beta),\ p \neq -1, 0.$$

Now we shall use the results of Section 1 to prove the following new facts for the above means.

PROPOSITION 1. *Let $p \geq 1$ and $0 \leq a \leq b$. Then, the following hold*

$$0 \leq A(a^p, b^p) - L_p^p(a,b) \leq \frac{p}{2(p-1)} \frac{\left[L_{p-1}^{p-1}(a,b) - a^{p-1}\right]\left[b^{p-1} - L_{p-1}^{p-1}(a,b)\right]}{L_{p-2}^{p-2}(a,b)}$$
$$\leq \frac{p(p-1)}{8}(b-a)^2 L_{p-2}^{p-2}(a,b).$$

PROOF. By Corollary 4 applied to the convex mapping $f(x) = x^p$, $p \geq 1$, we have

$$\begin{aligned} 0 &\leq \frac{a^p+b^p}{2} - \frac{1}{b-a}\int_a^b x^p\,dx \\ &\leq \frac{\left[b^p - a^p - pa^{p-1}(b-a)\right]\left[pb^{p-1}(b-a) - b^p + a^p\right]}{2p\left(b^{p-1}-a^{p-1}\right)(b-a)} \\ &\leq \frac{p\left(b^{p-1}-a^{p-1}\right)(b-a)}{8}. \end{aligned}$$

Proposition 1 now follows from the facts that

$$b^p - a^p = p(b-a)L_{p-1}^{p-1}(a,b),$$

and

$$b^{p-1} - a^{p-1} = (p-1)(b-a)L_{p-2}^{p-2}(a,b).$$

PROPOSITION 2. *Let $0 < a < b$. Then, the following hold*

$$0 \leq L(a,b) - H(a,b) \leq L(a,b)\left(\frac{b-a}{a+b}\right)^2 \leq L(a,b)\frac{(b-a)^2}{4ab}. \tag{9}$$

PROOF. By Corollary 4 applied to the convex mapping $f(x) = 1/x$ on $[a,b] \subset (0,\infty)$, we have

$$\begin{aligned} 0 &\leq \frac{1/a+1/b}{2} - \frac{\ln b - \ln a}{b-a} \\ &\leq \frac{\left[1/b - 1/a + ((b-a)/a^2)\right]\left[-((b-a)/b^2) - 1/b + 1/a\right]}{2\left(1/a^2 - 1/b^2\right)(b-a)} = R, \qquad \text{(say)} \\ &\leq \frac{\left(1/a^2 - 1/b^2\right)(b-a)}{8}. \end{aligned}$$

However, since

$$\frac{1}{b} - \frac{1}{a} + \frac{b-a}{a^2} = \frac{(b-a)^2}{a^2 b},$$

and

$$-\frac{b-a}{b^2} - \frac{1}{b} + \frac{1}{a} = \frac{(b-a)^2}{ab^2},$$

it follows that

$$R = \frac{(b-a)^2}{2ab(a+b)}.$$

Consequently, we get

$$0 \leq H^{-1}(a,b) - L^{-1}(a,b) \leq \frac{(b-a)^2}{2ab(a+b)} \leq \frac{(b-a)^2(a+b)}{8a^2b^2},$$

which is equivalent to (9).

PROPOSITION 3. *Let $0 < a < b$. Then, the following holds*

$$0 \leq \ln I(a,b) - \ln G(a,b) \leq \frac{ab\left[\ln(a/b) + ((b-a)/a)\right]\left[\ln(b/a) - ((b-a)/b)\right]}{2(b-a)^2} \leq \frac{(b-a)^2}{8ab}.$$

PROOF. The proof follows from Corollary 4 applied to the convex mapping $f(x) = -\ln x$.

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